

Linear Quaternion Differential Equations: Basic Theory and Fundamental Results (II)

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Abstract

The theory of two-dimensional linear Quaternion-valued differential equations (QDEs) was recently established. Some profound differences between QDEs and ordinary differential equations (ODEs) were observed. The solutions set of the QDEs is actually a right free module, not a linear vector space. An algorithm to evaluate the fundamental matrix by employing the eigenvalues and eigenvectors was recently presented. However, the fundamental matrix can be constructed providing that the eigenvalues are simple. If the linear system has multiple eigenvalues, how to construct the fundamental matrix? In particular, if the number of independent eigenvectors might be less than the dimension of the system. That is, the numbers of the eigenvectors is not enough to construct a fundamental matrix. How to find the “missing solutions”? The main purpose of this paper is to answer this question.

Furthermore, Caley determinant for Quaternion-valued matrix was adopted to proceed the theory of QDEs. One big disadvantage of Caley determinant is that it can be expanded along the different rows and columns. This may lead to different results due to non-commutativity of the quaternions. This approach is not convenient to be used. The novel definition of determinant for Quaternion-valued matrix based on permutation is introduced to analyze the theory. This newly definition of determinant has great advantage compare to Calay determinant.

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1 Introduction and Motivation

Quaternion-valued differential equations (QDEs) is a new kind of differential equations which have many applications in quantum mechanics (see e.g. [1, 2, 13, 14]), fluid mechanics(e.g. [9, 10, 17, 18]), Frenet frame in differential geometry (see e.g.[11]), kinematic modelling, attitude dynamics (see e.g. [28]), Kalman filter design and spatial rigid body dynamics ([25, 27]), etc. Though there are many applications in physics and life science, there are few papers pursuing mathematical analysis for the QDEs. Leo and Ducati [15] tried to solve some simple second order quaternionic differential equations. Campos and Mawhin [7] studied the existence of periodic solutions of one-dimensional first order periodic quaternion differential equation. Wilczynski [19] proved the existence of two periodic solutions of quaternion Riccati equations. As $n = 2, 3$, Gasull et al [8] proved the existence of periodic orbits, homoclinic loops, invariant tori for a one-dimensional quaternion autonomous homogeneous differential equation

$$\dot{q} = aq^n.$$

Later, Zhang [20] studied the global structure of the one-dimensional quaternion Bernoulli equations

$$\dot{q} = aq + aq^n.$$

Recently, Kou and Xia [12] presented a basic theory for the two-dimensional linear QDEs. There are some profound differences between QDEs and ODEs. The largest difference between QDEs and ODEs is the algebraic structure. On the non-commutativity of the quaternion algebra, the algebraic structure of the solutions to the QDEs is completely different from ODEs. It is actually a right-free module, not a linear vector space. Another big difference is that it is necessary to treat the eigenvalue problems with left- and right-sides, accordingly. In [12], the authors presented an algorithm to evaluate the fundamental matrix by employing the eigenvalue and eigenvectors. They provided a method and an example to show how to construct the fundamental matrix when the eigenvalues are simple. However, it is possible to have multiple eigenvalues. How can one construct the fundamental matrix when the multiplicity of the eigenvalues is larger than one? In particular, if the number of independent eigenvectors might be smaller than the dimensionality of the system. That is, the numbers of the eigenvectors is not enough to construct a fundamental matrix. In this paper, one of the main tasks is to discover the “missing solutions”. We will devote ourselves to answer this question.

On the other hand, the theory of n dimensional linear QDEs is more complicate. Different definitions of are not the same. Cayley determinant for quaternion valued matrix [6] was adopted in [12] which depends on the expansion of i -th row and j -th column of the quaternion-valued matrix, different expansions can lead to different results. For instance, considering a 2×2 determinant of quaternion valued matrix, we could use $a_{11}a_{22} - a_{12}a_{21}$ (expanding along the first row). That is,

$$\text{rdet} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Certainly, one can also use $a_{11}a_{22} - a_{21}a_{12}$ (expanding along the first column) as follows.

$$\text{cdet} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

There are other definitions (expanding along the second column) or (expanding along the second row). Owing to the non-commutativity of the quaternion algebra, the results are distinct due to different expansions. Therefore Cayley determinant is not convenient for the quaternion valued matrix. We will adopt another definition (see e.g. Chen [21]) to analyze our results in this paper. This definition of determinant which is based on permutation has great advantage compared to Cayley determinant (see next section in detail). Due to the newly definition of determinant, the computation of the determinant is different. In particular, the proof of Liouville formula is more complicated.

2 Quaternion algebra

The quaternions were discovered in 1843 by Sir William Rowan Hamilton [24]. We adopt the standard notation in [21, 24]. We denote as usual a quaternion $q = (q_0, q_1, q_2, q_3)^T \in \mathbb{R}^4$ by

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where q_0, q_1, q_2, q_3 are real numbers and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ symbols satisfy the multiplication table formed by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1.$$

The set of quaternion is denoted by \mathbb{H} .

If a quaternion in \mathbb{H} is $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, then its *conjugate* is $\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$. For any $p, h \in \mathbb{H}$, we have

$$\overline{qh} = \bar{h}\bar{q},$$

$$|q| = \sqrt{\bar{q}q} = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

Hence $q^{-1} = \frac{\bar{q}}{|q|^2}$, $q \neq 0$.

$$\Re q = q_0, \quad \Im q = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}.$$

Consider column n-vectors over \mathbb{H}

$$\alpha = (a_1, a_2, \dots, a_n)^\top, \beta = (b_1, b_2, \dots, b_n)^\top, a_i, b_i \in \mathbb{H}.$$

where α^\top is the transpose of α . Operations are given as

$$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)^\top, \alpha k = (a_1 k, a_2 k, \dots, a_n k)^\top, k \in \mathbb{H}.$$

and the inner product

$$(\alpha, \beta) = \bar{\alpha}^\top \beta = \alpha^+ \beta = \sum_{i=1}^n \bar{a}_i b_i.$$

where α^+ is the conjugate transpose of α .

As pointed out in the introduction, Caley determinant can be expanded along the j -th column or the i -th row. Owing the non-commutativity of the quaternion algebra, the results of the determinant are not same due to different expansions. Thus, it is not convenient to apply this definition to the quaternion valued matrix. In 1991, Chen [21] gave us a “direct” definition by specifying a certain ordering of the factors in the $n!$ terms in the sum. In this sense, the determinant has a unique result. So, in this article, we will study the n dimensional linear quaternionic-valued ordinary differential equations based on this definition. Now we are in a position to introduce this definition of determinant.

Let $\mathbb{H}^{n \times m}$ denote the set of all matrices $A = (a_{i,j})_{n \times m}$, where $a_{i,j}$ are quaternions. For any $A \in \mathbb{H}^{n \times n}$, the determinant based on permutation is defined as follows (see e.g. [21]).

$$\begin{aligned} \det_P A = |A|_P &= \det_P \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \\ &\equiv \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{n_1 i_2} a_{i_2 i_3} \cdots a_{i_{s-1} i_s} a_{i_s n_1} a_{n_2 j_2} \cdots a_{j_t n_2} \cdots a_{n_r k_2} \cdots a_{k_l n_r}, \end{aligned} \quad (2.1)$$

where S_n is the symmetric group on n letters, and the disjoint cycle decomposition of $\sigma \in S_n$ is written in the normal form:

$$\begin{aligned} \sigma &= (n_1 i_2 i_3 \cdots i_s)(n_2 j_2 j_3 \cdots j_t) \cdots (n_r k_2 k_3 \cdots k_l), \\ n_1 &> i_2, i_3, \cdots, i_s, n_2 > j_2, j_3, \cdots, j_t, \cdots, n_r > k_2, k_3, \cdots, k_l, \\ n &= n_1 > n_2 > \cdots > n_r \geq 1, \end{aligned}$$

and

$$\varepsilon(\sigma) = (-1)^{(s-1)+(t-1)+\cdots+(l-1)} = (-1)^{n-r}.$$

For sake of making difference from the ordinary determinant \det , we denote this determinant based on permutation \det_P . Notice that if all a_{ij} commute with each other, the definition of $\det A$ is the same as that an ordinary determinant (Caley determinant).

For convenience and explicitness, if cycle factor $\sigma_0 = (i_1 i_2 \cdots i_s)$, and we denote

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{s-1} i_s} = \langle i_1 i_2 \cdots i_s \rangle,$$

and

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{s-1} i_s} a_{i_s i_1} = \langle i_1 i_2 \cdots i_s i_1 \rangle.$$

Then expression Eq.(2.1) is simplified into

$$\det_P A = \sum_{\sigma \in S_n, \sigma = \sigma_1 \sigma_2 \cdots \sigma_r} \varepsilon(\sigma) \langle \sigma_1 \rangle \langle \sigma_2 \rangle \cdots \langle \sigma_r \rangle = \sum_{\sigma \in S_n} \varepsilon(\sigma) \langle \sigma \rangle.$$

In particular, for $n = 2$,

$$\sigma_1 = (2)(1), \sigma_2 = (21) \in S_2,$$

and

$$\varepsilon(\sigma_1) = (-1)^{(1-1)+(1-1)} = 1, \varepsilon(\sigma_2) = (-1)^{(2-1)} = -1.$$

So we get

$$\det_P \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \varepsilon(\sigma_1) a_{22} a_{11} + \varepsilon(\sigma_2) a_{21} a_{12} = a_{22} a_{11} - a_{21} a_{12}.$$

For $n = 3$.

$$\sigma_1 = (3)(2)(1), \sigma_2 = (3)(21), \sigma_3 = (312),$$

$$\sigma_4 = (31)(2), \sigma_5 = (321), \sigma_6 = (32)(1) \in S_3,$$

and

$$\begin{aligned} \varepsilon(\sigma_1) &= (-1)^{(3-1)} = 1, \varepsilon(\sigma_2) = (-1)^{(2-1)} = -1, \varepsilon(\sigma_3) = (-1)^{(1-1)} = 1 \\ \varepsilon(\sigma_4) &= (-1)^{(2-1)} = -1, \varepsilon(\sigma_5) = (-1)^{(1-1)} = -1, \varepsilon(\sigma_6) = (-1)^{(2-1)} = 1. \end{aligned}$$

So we get

$$\begin{aligned} & \det_P \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \varepsilon(\sigma_1)a_{33}a_{22}a_{11} + \varepsilon(\sigma_2)a_{33}a_{21}a_{12} + \varepsilon(\sigma_3)a_{31}a_{12}a_{23} + \varepsilon(\sigma_4)a_{31}a_{13}a_{22} + \varepsilon(\sigma_5)a_{32}a_{21}a_{13} + \varepsilon(\sigma_6)a_{32}a_{23}a_{11}, \\ &= a_{33}a_{22}a_{11} - a_{33}a_{21}a_{12} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22} + a_{32}a_{21}a_{13} - a_{32}a_{23}a_{11}. \end{aligned}$$

For any $A \in \mathbb{H}^{n \times m}$, $\text{ddet} A \equiv \det_P(A^+ A)$ is called the double determinant of A . Since $A^+ A$ is a Hermitian matrix, $\text{ddet} A$ is always a real number (it can be proved that $\text{ddet} A \geq 0$).

Let $\psi : \mathbb{R} \rightarrow \mathbb{H}$ be a quaternion-valued function defined on \mathbb{R} ($t \in \mathbb{R}$ is a real variable). We denote the set of such quaternion-valued functions by $\mathbb{H} \otimes \mathbb{R}$. Then n dimensional quaternionic functions of real variable, $\mathbb{H}^n \otimes \mathbb{R} = \{\Psi(t) | \Psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_n(t))^T\}$. And the derivative, integral and norms of n dimensional quaternionic functions with respect to the real variable t are well defined, one can refer to [12]. Moreover, we adopt these notations from [12].

3 Wronskian and Structure of General Solution to QDEs

Consider the linear QDEs as follows.

$$\dot{\Psi}(t) = A(t)\Psi(t) \quad \text{or} \quad \begin{pmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \\ \dots \\ \dot{\psi}_n(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \cdots a_{1n}(t) \\ a_{21}(t) & a_{22}(t) \cdots a_{2n}(t) \\ \dots & \dots \\ a_{n1}(t) & a_{n2}(t) \cdots a_{nn}(t) \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \dots \\ \psi_n(t) \end{pmatrix}, \quad (3.1)$$

where $\Psi(t) \in \mathbb{H}^n \otimes \mathbb{R}$, $A(t) \in \mathbb{H}^{n \times n} \otimes \mathbb{R}$ is continuous on the interval $[a, b]$.

Considering systems (3.1) are associated with the following initial value problem

$$\Psi(t_0) = \xi, \quad \xi \in \mathbb{H}^n. \quad (3.2)$$

Similar discussion to Theorem 3.1 in [12], we have

Theorem 3.1. *The initial value problem (3.1)-(3.2) has exactly one solution.*

Now we introduce some definitions on abelian groups, rings, modules, submodules, direct sum in the abstract algebraic theory (e.g. [35, 36]). Due to the length of this paper, we omit the definitions. For these detailed definitions, one can refer to [12]. We also adopt the notations from [12]. An *abelian group* is a set denoted by \mathbb{A} . A *ring* is a set denoted by $(\mathcal{R}, +, \cdot)$ equipped with binary operations $+$ and \cdot . $(\mathcal{F}, +, \cdot)$ is said to be a field if $(\mathcal{F}, +, \cdot)$ is a ring and $\mathcal{F}^* = \mathcal{F}/\{0\}$ with respect to multiplication \cdot is a commutative group. We will write $\mathbb{A}_{\mathcal{R}}^R$ to indicate that \mathbb{A} is a right \mathcal{R} -module (over ring \mathcal{R}). Similarly, we can define the left \mathcal{R} -module. And we denote the left \mathcal{R} -module by $\mathbb{A}_{\mathcal{R}}^L$ (over ring \mathcal{R}). The elements $x_1, \dots, x_k \in \mathbb{A}_{\mathcal{R}}^R$ are called independent if

$$x_1 r_1 + \dots + x_k r_k = 0, r_i \in \mathcal{R} \text{ implies that } r_1 = \dots = r_k = 0,$$

that is, the only linear combination that vanishes is the trivial one with every coefficient zero. For the sake of convenience, we call it *right independence*. A subset $\{x_1, \dots, x_k\}$ is called a basis of a right module $\mathbb{A}_{\mathcal{R}}^R$ if it is right independent and generates $\mathbb{A}_{\mathcal{R}}^R$. A module that has a finite basis is called a **free module**.

Direct differentiation, one can verify that

Theorem 3.2. (*superposition theorem*) If $u(t)$ and $v(t)$ are two solutions of Eq.(3.1), then the linear combination $u(t)\alpha + v(t)\beta$ is also a solution of Eq.(3.1), where $\alpha, \beta \in \mathbb{H}$.

By Theorem 3.2, it is possible that all solutions of Eq.(3.1) can generate the right \mathbb{H} -module. We claim that the set of all the solutions of Eq.(3.1) is the right \mathbb{H} -module. To prove this, firstly we try to find a basis of this right \mathbb{H} -module. Now we should introduce the concept of independence and dependence for the vector functions $x_1(t), x_2(t), \dots, x_n(t)$.

Definition 3.3. For n quaternion-valued vector functions $x_1(t), x_2(t), \dots, x_n(t)$, each $x_i(t) \in \mathbb{H}^n \otimes \mathbb{R}$ defined on the real interval I , if

$$x_1(t)r_1 + \dots + x_n(t)r_n = 0, r_i \in \mathbb{H} \text{ implies that } r_1 = \dots = r_n = 0, t \in I$$

then $x_1(t), x_2(t), \dots, x_n(t)$ is said to be independent. Otherwise, $x_1(t), x_2(t), \dots, x_n(t)$ is said to be dependent.

Definition 3.4. Let $x_1(t), x_2(t), \dots, x_n(t)$ are n solutions of Eq.(3.1). Denote

$$M(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \dots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \dots & \dots & \dots & \dots \\ x_{n1}(t) & x_{n2}(t) & \dots & x_{nn}(t) \end{pmatrix}.$$

The Wronskian of QDEs is defined by

$$W_{QDE}(t) = \frac{1}{2} \text{ddet} M(t) := \frac{1}{2} \det_{\mathbb{P}} (M^+(t)M(t))$$

where M^+ is the conjugate transpose of $M(t)$, namely

$$M^+(t) = \begin{pmatrix} \bar{x}_{11}(t) & \bar{x}_{21}(t) \cdots \bar{x}_{n1}(t) \\ \bar{x}_{12}(t) & \bar{x}_{22}(t) \cdots \bar{x}_{n2}(t) \\ & \dots\dots\dots \\ \bar{x}_{1n}(t) & \bar{x}_{2n}(t) \cdots \bar{x}_{nn}(t) \end{pmatrix}.$$

Remark 3.5. As pointed out in [12], the standard Wronskian of ODEs is not valid for QDEs. So we define Wronskian of QDEs by $\frac{1}{2} \det_P (M^+(t)M(t))$. For quaternion matrix, it should be noted that

$$\det_P (A(t)B(t)) \neq \det_P A(t) \cdot \det_P B(t).$$

But, in [22] (Theorem 5), he proved that

$$\text{ddet}(A(t)B(t)) = \text{ddet}A(t) \cdot \text{ddet}B(t).$$

Theorem 3.6. If $x_1(t), x_2(t), \dots, x_n(t)$ are right dependent on I , then $W_{QDE}(t) = 0$.

Proof. If $x_1(t), x_2(t), \dots, x_n(t)$ are right dependent on I , then there exist $r_1, r_2, \dots, r_n \in \mathbb{H}$ (not all zero) such that

$$x_1(t)r_1 + x_2(t)r_2 + \cdots + x_n(t)r_n = 0, \quad t \in I,$$

or

$$(x_1(t), x_2(t), \dots, x_n(t)) \begin{pmatrix} r_1 \\ r_2 \\ \dots \\ r_n \end{pmatrix} = 0, \quad t \in I \quad (3.3)$$

In Eq.(3.3), $(x_1(t), x_2(t), \dots, x_n(t))$ can be seen as the coefficient matrix, r_1, r_2, \dots, r_n can be seen as unknown numbers. Thus, Eq.(3.3) can be seen as homogeneous system of right linear equations. By way of contradiction, if there exists $t_0 \in I$ such that $W_{QDE}(t_0) \neq 0$, by Theorem 2.3 [21], then exists a unique solution $r_1 = r_2 = \cdots = r_n = 0$. This implies that $x_1(t), x_2(t), \dots, x_n(t)$ are right dependent on I , which is a contradiction. Therefore, $W_{QDE}(t) = 0, t \in I$.

We need a lemma from Theorem 8 in [22].

Lemma 3.7. Let $A_{n \times m} = (\alpha_1, \alpha_2, \dots, \alpha_m)^T \in \mathbb{H}^{n \times m}$, where $\alpha_1, \alpha_2, \dots, \alpha_m$ are m column quaternionic vectors. Then $\alpha_1, \alpha_2, \dots, \alpha_m$ are right independent if and only if $\text{ddet}A_{n \times m} \neq 0$.

In particular, for $n = m$, Then $\alpha_1, \alpha_2, \dots, \alpha_n$ are right independent if and only if $\text{ddet}A_{n \times n} \neq 0$.

Theorem 3.8. If $x_1(t), x_2(t), \dots, x_n(t)$ are n right independent solutions of Eq.(3.1), then $W_{QDE}(t) \neq 0$ on I .

Proof. By way of contradiction. We assume that there exists $t_0 \in I$ such that $W_{QDE}(t_0) = 0$. Let us consider the homogeneous system of right linear equations as follows

$$x_1(t_0)r_1 + x_2(t_0)r_2 + \cdots + x_n(t_0)r_n = 0, \quad t_0 \in I. \quad (3.4)$$

We claim that there exists n quaternionic constants $\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n \in \mathbb{H}$ (not all zero) satisfying the Eq.(3.4). Or else, $r_1 = r_2 = \cdots = r_n = 0$, which implies that $x_1(t_0), x_2(t_0), \dots, x_n(t_0)$ are n independent. By Lemma 3.7, $W_{QDE}(t_0) \neq 0$, which is contradicted to our assumption $W_{QDE}(t_0) = 0$. Thus, the claim is true. Based on the nonzero quaternionic constants $\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n \in \mathbb{H}$, we construct a new function as below

$$x(t) = x_1(t)\tilde{r}_1 + x_2(t)\tilde{r}_2 + \cdots + x_n(t)\tilde{r}_n, \quad t \in I.$$

Then it is easy to verify that $x(t)$ is a solutions of Eq.(3.1) and satisfies initial condition

$$x(t_0) = 0 \quad (3.5)$$

Obviously, $x(t) \equiv 0$ is also a solutions of Eq.(3.1) and initial condition Eq.(3.5). By the uniqueness of the solutions to the same initial value, we have

$$x(t) = x_1(t)\tilde{r}_1 + x_2(t)\tilde{r}_2 + \cdots + x_n(t)\tilde{r}_n = 0, \quad t \in I,$$

which implies that $x_1(t), x_2(t), \dots, x_n(t)$ are right dependent (due to $\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n \in \mathbb{H}$ are nonzero). This is contradictory to $x_1(t), x_2(t), \dots, x_n(t)$ are right independent on I . Therefore, $W_{QDE}(t) \neq 0$ on I .

Theorem 3.9. (*Liouville formula*) The Wronskian $W_{QDE}(t)$ of Eq. (3.1) satisfies the following quaternionic Liouville formula.

$$W_{QDE}(t) = \exp\left(\frac{1}{2} \int_{t_0}^t [\text{tr}A(s) + \text{tr}A^+(s)]ds\right) W_{QDE}(t_0),$$

or

$$W_{QDE}(t) = \exp\left(\int_{t_0}^t \Re(\text{tr}A(t))ds\right) W_{QDE}(t_0),$$

where $\text{tr}A(t)$ is the trace of the coefficient matrix $A(t)$, i.e. $\text{tr}A(t) = \sum_{i=1}^n a_{ii}(t)$. Moreover, if $W_{QDE}(t) = 0$ at some t_0 in I then $W_{QDE}(t) = 0$ on I .

Remark 3.10. In previous work, we prove the Liouville formula based on Caley determinant. We employ the definition of determinant based on permutation in this paper. The proof is more complicate.

Proof. By definition of determinant

$$\frac{d}{dt}W_{QDE}(t) = \frac{1}{2} \frac{d}{dt} \left| \begin{pmatrix} \bar{x}_{11}(t) & \bar{x}_{21}(t) & \cdots & \bar{x}_{n1}(t) \\ \bar{x}_{12}(t) & \bar{x}_{22}(t) & \cdots & \bar{x}_{n2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{x}_{1n}(t) & \bar{x}_{2n}(t) & \cdots & \bar{x}_{nn}(t) \end{pmatrix} \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix} \right|_P$$

$$\begin{aligned}
&= \frac{1}{2} \frac{d}{dt} \left| \begin{array}{cccc} \sum_{i=1}^n \bar{x}_{i1}(t)x_{i1}(t) & \sum_{i=1}^n \bar{x}_{i1}(t)x_{i2}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i1}(t)x_{in}(t) \\ \sum_{i=1}^n \bar{x}_{i2}(t)x_{i1}(t) & \sum_{i=1}^n \bar{x}_{i2}(t)x_{i2}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i2}(t)x_{in}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n \bar{x}_{in}(t)x_{i1}(t) & \sum_{i=1}^n \bar{x}_{in}(t)x_{i2}(t) & \cdots & \sum_{i=1}^n \bar{x}_{in}(t)x_{in}(t) \end{array} \right|_P \\
&= \frac{1}{2} \left| \begin{array}{cccc} \frac{d}{dt} \sum_{i=1}^n \bar{x}_{i1}(t)x_{i1}(t) & \cdots & \frac{d}{dt} \sum_{i=1}^n \bar{x}_{i1}(t)x_{in}(t) \\ \sum_{i=1}^n \bar{x}_{i2}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i2}(t)x_{in}(t) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \bar{x}_{in}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{in}(t)x_{in}(t) \end{array} \right|_P + \frac{1}{2} \left| \begin{array}{cccc} \sum_{i=1}^n \bar{x}_{i1}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i1}(t)x_{in}(t) \\ \frac{d}{dt} \sum_{i=1}^n \bar{x}_{i2}(t)x_{i1}(t) & \cdots & \frac{d}{dt} \sum_{i=1}^n \bar{x}_{i2}(t)x_{in}(t) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \bar{x}_{in}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{in}(t)x_{in}(t) \end{array} \right|_P \\
&\quad + \cdots + \frac{1}{2} \left| \begin{array}{cccc} \sum_{i=1}^n \bar{x}_{i1}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i1}(t)x_{in}(t) \\ \sum_{i=1}^n \bar{x}_{i2}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i2}(t)x_{in}(t) \\ \vdots & \ddots & \vdots \\ \frac{d}{dt} \sum_{i=1}^n \bar{x}_{in}(t)x_{i1}(t) & \cdots & \frac{d}{dt} \sum_{i=1}^n \bar{x}_{in}(t)x_{in}(t) \end{array} \right|_P
\end{aligned} \tag{3.6}$$

Since $x_1(t), x_2(t), \dots, x_k(t)$, $k = 1, 2, \dots, n$ are n solutions of the Eq.(3.1). Therefore

$$\begin{aligned}
&\left| \begin{array}{cccc} \frac{d}{dt} \sum_{i=1}^n \bar{x}_{i1}(t)x_{i1}(t) & \cdots & \frac{d}{dt} \sum_{i=1}^n \bar{x}_{i1}(t)x_{in}(t) \\ \sum_{i=1}^n \bar{x}_{i2}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i2}(t)x_{in}(t) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \bar{x}_{in}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{in}(t)x_{in}(t) \end{array} \right|_P = \\
&\left| \begin{array}{cccc} \sum_{i=1}^n \sum_{j=1}^n (\bar{x}_{j1}(t)\bar{a}_{ij}(t)x_{i1}(t) + \bar{x}_{i1}(t)a_{ij}(t)x_{j1}(t)) & \cdots & \sum_{i=1}^n \sum_{j=1}^n (\bar{x}_{j1}(t)\bar{a}_{ij}(t)x_{in}(t) + \bar{x}_{i1}(t)a_{ij}(t)x_{jn}(t)) \\ \sum_{i=1}^n \bar{x}_{i2}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i2}(t)x_{in}(t) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \bar{x}_{in}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{in}(t)x_{in}(t) \end{array} \right|_P
\end{aligned}$$

$$\begin{aligned}
&= \left| \begin{array}{cccc} 2\Re a_{ii} \sum_{i=1}^n \bar{x}_{i1}(t)x_{i1}(t) & 2\Re a_{ii} \sum_{i=1}^n \bar{x}_{i1}(t)x_{i2}(t) & \cdots & 2\Re a_{ii} \sum_{i=1}^n \bar{x}_{i1}(t)x_{in}(t) \\ \sum_{i=1}^n \bar{x}_{i2}(t)x_{i1}(t) & \sum_{i=1}^n \bar{x}_{i2}(t)x_{i2}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i2}(t)x_{in}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n \bar{x}_{in}(t)x_{i1}(t) & \sum_{i=1}^n \bar{x}_{in}(t)x_{i2}(t) & \cdots & \sum_{i=1}^n \bar{x}_{in}(t)x_{in}(t) \end{array} \right|_P \\
&+ \sum_{k=1}^n \sum_{\substack{j=1 \\ j \neq k}}^n \left| \begin{array}{ccc} \bar{x}_{j1}(t)\bar{a}_{kj}(t)x_{k1}(t) + \bar{x}_{k1}(t)a_{kj}(t)x_{j1}(t) \cdots \bar{x}_{j1}(t)\bar{a}_{kj}(t)x_{in}(t) + \bar{x}_{k1}(t)a_{kj}(t)x_{jn}(t) \\ \sum_{i=1}^n \bar{x}_{i2}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{i2}(t)x_{in}(t) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \bar{x}_{in}(t)x_{i1}(t) & \cdots & \sum_{i=1}^n \bar{x}_{in}(t)x_{in}(t) \end{array} \right|_P \\
&:= A_1(t) + \sum_{k=1}^n \sum_{\substack{j=1 \\ j \neq k}}^n B_{kj}^1(t).
\end{aligned}$$

Similarly, the expansion of the m determinant of the Eq.(3.6) is $A_m(t) + \sum_{k=1}^n \sum_{\substack{j=1 \\ j \neq k}}^n B_{kj}^m(t)$. ($m = 1, 2, \dots, n$). So we get

$$\frac{d}{dt} W_{QDE}(t) = \frac{1}{2} \sum_{m=1}^n A_m(t) + \frac{1}{2} \sum_{m=1}^n \sum_{k=1}^n \sum_{\substack{j=1 \\ j \neq k}}^n B_{kj}^m(t) = \frac{1}{2} \sum_{m=1}^n A_m(t) + \frac{1}{2} \sum_{k=1}^n \sum_{\substack{j=1 \\ j \neq k}}^n \sum_{m=1}^n B_{kj}^m(t). \quad (3.7)$$

(i) Considering $W_{QDE}(t)$. For every permutation $\sigma \in S_n$,

$$\sigma = (n_1 i_2 \cdots i_s) \cdots (n_p j_2 \cdots j_q) \cdots (n_r k_2 \cdots k_t) = \sigma_1 \cdots \sigma_p \cdots \sigma_r,$$

we have

$$\begin{aligned}
\langle \sigma \rangle &= \left(\sum_{i=1}^n \bar{x}_{in_1} x_{ii_2} \right) \cdots \left(\sum_{i=1}^n \bar{x}_{ii_s} x_{in_1} \right) \cdots \left(\sum_{i=1}^n \bar{x}_{in_p} x_{ij_2} \right) \cdots \\
&\quad \left(\sum_{i=1}^n \bar{x}_{ij_q} x_{in_p} \right) \cdots \left(\sum_{i=1}^n \bar{x}_{in_r} x_{ik_2} \right) \cdots \left(\sum_{i=1}^n \bar{x}_{ik_t} x_{in_r} \right)
\end{aligned} \quad (3.8)$$

where for sake of convenience, $x(t)$ is briefly denoted by x . If we expand (3.8), every expanded term can be expressed as

$$\bar{x}_{i_{n_1} n_1} x_{i_{n_1} i_2} \cdots \bar{x}_{i_{i_s} i_s} x_{i_{i_s} n_1} \cdots \bar{x}_{i_{n_p} n_p} x_{i_{n_p} j_2} \cdots \bar{x}_{i_{j_q} j_q} x_{i_{j_q} n_p} \cdots \bar{x}_{i_{n_r} n_r} x_{i_{n_r} k_2} \cdots \bar{x}_{i_{k_t} k_t} x_{i_{k_t} n_r} \cdots,$$

where $\bar{x}_{i_{n_1} n_1} x_{i_{n_1} i_2}$ is the i_{n_1} form of $\sum_{i=1}^n \bar{x}_{in_1} x_{ii_2}$ and similar for the rest.

If $i_{n_1} = i_{i_2}$, considering the product

$$\bar{x}_{i_{n_1} n_1} x_{i_{n_1} i_2} \bar{x}_{i_{i_2} i_2} x_{i_{i_2} i_3} \cdots = (x_{i_{n_1} i_2} \bar{x}_{i_{i_2} i_2}) \bar{x}_{i_{n_1} n_1} x_{i_{i_2} i_3} \cdots, \quad (3.9)$$

which is one of terms of $\langle \sigma \rangle$, we have

$$\sigma_1^* = (n_1 i_3 \cdots i_s), \sigma_1^\nabla = (i_2),$$

and

$$\sigma^\nabla = \sigma_1^* \cdots \sigma_1^\nabla \cdots$$

We see that

$$\overline{x}_{i_{n_1} n_1} x_{i_{i_2} i_3} \cdots x_{i_{n_1} i_2} \overline{x}_{i_{i_2} i_2} \cdots \quad (3.10)$$

is a term of $\langle \sigma^\nabla \rangle$. It is obvious that (3.9) is equivalent to (3.10) and $\varepsilon(\sigma) = -\varepsilon(\sigma^\nabla)$. Thus, the term (3.9) and the term (3.10) canceled each other out when $i_{n_1} = i_{i_2}$. After the terms with $i_{n_1} = i_{i_2}$ canceled out, in the rest terms, i_{n_1} is different from i_{i_2} .

If $i_{n_1} = i_{i_\omega}$, $i_\omega \in \{i_3, \cdots, i_s\}$. For the product

$$\overline{x}_{i_{n_1} n_1} x_{i_{i_\omega} i_\omega} \overline{x}_{i_{i_2} i_2} x_{i_{i_2} i_3} \cdots \overline{x}_{i_{i_{\omega-1}} i_{\omega-1}} x_{i_{i_{\omega-1}} i_\omega} \overline{x}_{i_{i_\omega} i_\omega} x_{i_{i_\omega} i_{\omega+1}} \overline{x}_{i_{i_{\omega+1}} i_{\omega+1}} x_{i_{i_{\omega+1}} i_{\omega+2}} \cdots, \quad (3.11)$$

there should be a corresponding term which can be expressed as follows

$$\overline{x}_{i_{n_1} n_1} x_{i_{i_\omega} i_\omega} \overline{x}_{i_{i_{\omega-1}} i_{\omega-1}} x_{i_{i_{\omega-1}} i_{\omega-1}} \cdots \overline{x}_{i_{i_2} i_3} x_{i_{i_2} i_2} \overline{x}_{i_{n_1} i_2} x_{i_{i_\omega} i_{\omega+1}} \overline{x}_{i_{i_{\omega+1}} i_{\omega+1}} x_{i_{i_{\omega+1}} i_{\omega+2}} \cdots \quad (3.12)$$

is one term of $\langle \sigma^\diamond \rangle$

$$\sigma^\diamond = (n_1, i_\omega, i_{\omega-1}, \cdots, i_3, i_2, i_{\omega+1}, \cdots, i_s) \cdots (n_p, j_2, \cdots, j_q) \cdots (n_r, k_2, \cdots, k_t) = \sigma_1^\diamond \cdots \sigma_p \cdots \sigma_r,$$

and

$$x_{i_{n_1} i_2} \overline{x}_{i_{i_2} i_2} x_{i_{i_2} i_3} \cdots \overline{x}_{i_{i_{\omega-1}} i_{\omega-1}} x_{i_{i_{\omega-1}} i_\omega} \overline{x}_{i_{i_\omega} i_\omega} = \overline{x_{i_{i_\omega} i_\omega} \overline{x}_{i_{i_{\omega-1}} i_{\omega-1}} x_{i_{i_{\omega-1}} i_\omega} \cdots \overline{x}_{i_{i_2} i_3} x_{i_{i_2} i_2} \overline{x}_{i_{n_1} i_2}}.$$

Thus, the product (3.11) plus the product (3.12) (for the sake of convenience, denoted by (3.11) + (3.12)) leads to

$$(3.11) + (3.12) = 2\Re(x_{i_{n_1} i_2} \overline{x}_{i_{i_2} i_2} x_{i_{i_2} i_3} \cdots \overline{x}_{i_{i_{\omega-1}} i_{\omega-1}} x_{i_{i_{\omega-1}} i_\omega} \overline{x}_{i_{i_\omega} i_\omega}) \overline{x}_{i_{n_1} n_1} x_{i_{i_\omega} i_{\omega+1}} \cdots$$

Let i_m be the biggest number in a set $\{i_2, i_3, \cdots, i_{\omega-1}, i_\omega\}$, then we have

$$\sigma_1^* = (n_1 i_{\omega+1} \cdots i_s), \sigma_1^\nabla = (i_m i_{m+1} \cdots i_\omega i_2 i_3 \cdots i_{m-1}), \sigma_1^\Delta = (i_m i_{m-1} \cdots i_3 i_2 i_\omega \cdots i_{m+1}),$$

and

$$\sigma^\nabla = \sigma_1^* \cdots \sigma_1^\nabla \cdots, \sigma^\Delta = \sigma_1^* \cdots \sigma_1^\Delta \cdots$$

We see that

$$\overline{x}_{i_{n_1} n_1} x_{i_{i_\omega} i_{\omega+1}} \overline{x}_{i_{i_{\omega+1}} i_{\omega+1}} x_{i_{i_{\omega+1}} i_{\omega+2}} \cdots \overline{x}_{i_{i_m} i_m} x_{i_{i_m} i_{m+1}} \cdots \overline{x}_{i_{i_\omega} i_\omega} x_{i_{i_\omega} i_2} \cdots \overline{x}_{i_{i_{m-1}} i_{m-1}} x_{i_{i_{m-1}} i_m} \cdots \quad (3.13)$$

$$\overline{x}_{i_{n_1} n_1} x_{i_{i_\omega} i_{\omega+1}} \overline{x}_{i_{i_{\omega+1}} i_{\omega+1}} x_{i_{i_{\omega+1}} i_{\omega+2}} \cdots \overline{x}_{i_{i_m} i_m} x_{i_{i_m} i_{m-1}} \cdots \overline{x}_{i_{i_2} i_2} x_{i_{i_2} i_\omega} \cdots \overline{x}_{i_{i_{m+1}} i_{m+1}} x_{i_{i_{m+1}} i_m} \cdots \quad (3.14)$$

(3.13) is one term of $\langle \sigma^\nabla \rangle$ and (3.14) is one term of $\langle \sigma^\Delta \rangle$. In view of $\sigma_1^\nabla = \overline{\sigma_1^\Delta}$, we have

$$(3.13) + (3.14) = 2\Re(\overline{x}_{i_{i_m} i_m} x_{i_{i_m} i_{m+1}} \cdots \overline{x}_{i_{i_\omega} i_\omega} x_{i_{i_\omega} i_2} \cdots \overline{x}_{i_{i_{m-1}} i_{m-1}} x_{i_{i_{m-1}} i_m}) \overline{x}_{i_{n_1} n_1} x_{i_{i_\omega} i_{\omega+1}} \cdots$$

By $\Re(ab) = \Re(ba)$, a, b are quaternions. Therefore

$$\begin{aligned} & \Re(x_{i_{n_1}i_2}\bar{x}_{i_{i_2}i_2}x_{i_{i_2}i_3}\cdots\bar{x}_{i_{i_{\omega-1}}i_{\omega-1}}x_{i_{i_{\omega-1}}i_{\omega}}\bar{x}_{i_{i_{\omega}}i_{\omega}}) \\ &= \Re(\bar{x}_{i_{i_m}i_m}x_{i_{i_m}i_{m+1}}\cdots\bar{x}_{i_{i_{\omega}}i_{\omega}}x_{i_{i_{\omega}}i_2}\cdots\bar{x}_{i_{i_{m-1}}i_{m-1}}x_{i_{i_{m-1}}i_m}). \end{aligned}$$

It is easy to see that $\varepsilon(\sigma) = \varepsilon(\sigma^\diamond) = -\varepsilon(\sigma^\nabla) = -\varepsilon(\sigma^\triangle)$. Therefore, the terms (3.11) + (3.12) and (3.13) + (3.14) canceled each other out when $i_{n_1} = i_{i_\omega}$, $i_\omega \in \{i_3, \dots, i_s\}$. After such terms with $i_{n_1} = i_{i_\omega}$ canceled out, in the rest terms, i_{n_1} is different from $i_{i_2}, i_{i_3}, \dots, i_{i_s}, \dots, i_{n_p}, \dots, i_{j_q}, \dots, i_{n_r}, \dots, i_{k_l}$. The rest can be done in the same manner, i_{j_ω} is different from $i_{j_{\omega+1}}, i_{j_{\omega+2}}, \dots, i_{j_q}, \dots, i_{n_p}, \dots, i_{i_{j_q}}, \dots, i_{n_r}, \dots, i_{k_l}$. Finally we can prove that $i_{n_1}, i_{i_2}, i_{i_3}, \dots, i_{i_s}, \dots, i_{n_p}, \dots, i_{i_{j_q}}, \dots, i_{n_r}, \dots, i_{k_l}$ are distinct from each other in the rest terms. According to $\Re a_{ii} \in \mathbb{R}$. Obviously, $i_{n_1}, i_{i_2}, i_{i_3}, \dots, i_{i_s}, \dots, i_{n_p}, \dots, i_{i_{j_q}}, \dots, i_{n_r}, \dots, i_{k_l}$ are distinct from each other in the rest terms of $A_1(t), A_2(t), \dots, A_n(t)$.

For $W_{QDE}(t)$. Let

$$\bar{x}_{i_{n_1}n_1}x_{i_{n_1}i_2}\cdots\bar{x}_{i_{i_s}i_s}x_{i_{i_s}n_1}\cdots\bar{x}_{i_{n_p}n_p}x_{i_{n_p}j_2}\cdots\bar{x}_{i_{j_q}n_p}x_{i_{j_q}n_p}\cdots\bar{x}_{i_{n_r}n_r}x_{i_{n_r}k_2}\cdots\bar{x}_{i_{k_l}k_l}x_{i_{k_l}n_r} = a$$

be one term of the rest in $\langle \sigma \rangle$. And for the rest terms of $A_1(t), A_2(t), \dots, A_n(t)$, the coefficient of a must be distinct from each other. Therefore, by the arbitrariness of a and $\langle \sigma \rangle$, we easily get

$$\sum_i^n A_i(t) = \sum_i^n (2\Re a_{ii}a + \cdots) = \sum_i^n 2\Re a_{ii}(a + \cdots) = \sum_i^n 2\Re a_{ii}W_{QDE}(t). \quad (3.15)$$

(ii) For arbitrary k, j ($k \neq j$), we will prove $\sum_{m=1}^n B_{kj}^m(t) = 0$. Without loss of generality, let $k = 1, j = 2$. For every permutation $\sigma \in S_n$

$$\sigma = (n_1i_2 \cdots i_s) \cdots (n_pj_2 \cdots j_q) \cdots (n_rk_2 \cdots k_t) = \sigma_1 \cdots \sigma_p \cdots \sigma_r$$

Consider $B_{12}^{j_\omega}(t)$ and $\sigma_p = (n_pj_2 \cdots j_{\omega-1}j_\omega j_{\omega+1} \cdots j_q)$. Every expanded terms of $\langle \sigma \rangle$ can be expressed as

$$\bar{x}_{i_{n_1}n_1}x_{i_{n_1}i_2} \cdots \cdots \bar{x}_{i_{j_{\omega-1}}j_{\omega-1}}x_{i_{j_{\omega-1}}1}\bar{x}_{2j_\omega}\bar{a}_{12}x_{1j_{\omega+1}}\bar{x}_{i_{j_{\omega+1}}j_{\omega+1}}x_{i_{j_{\omega+1}}j_{\omega+2}} \cdots \cdots \bar{x}_{i_{k_l}k_l}x_{i_{k_l}n_r}, \quad (3.16)$$

or

$$\bar{x}_{i_{n_1}n_1}x_{i_{n_1}i_2} \cdots \cdots \bar{x}_{i_{j_{\omega-1}}j_{\omega-1}}x_{i_{j_{\omega-1}}1}\bar{x}_{1j_\omega}a_{12}x_{2j_{\omega+1}}\bar{x}_{i_{j_{\omega+1}}j_{\omega+1}}x_{i_{j_{\omega+1}}j_{\omega+2}} \cdots \cdots \bar{x}_{i_{k_l}k_l}x_{i_{k_l}n_r}. \quad (3.17)$$

From the proof of (i), we easily obtain that $i_{n_1}, \dots, i_{j_{\omega-1}}, i_{j_{\omega+2}}, \dots, i_{k_l}$ are distinct from each other in the rest terms.

Because $i_{n_1}, \dots, i_{j_{\omega-1}}, i_{j_{\omega+2}}, \dots, i_{k_l}$ are distinct from each other in the rest forms, there is $i_z = 1$ or $i_z = 2$ or $i_{z_1} = 1, i_{z_2} = 2$, $i_z, i_{z_1}, i_{z_2} \in \{i_{n_1}, \dots, i_{j_{\omega-1}}, i_{j_{\omega+2}}, \dots, i_{n_1}\}$.

For $i_z = 1$ or $i_z = 2$ or $i_{z_1} = 1, i_{z_2} = 2$, since (3.16) contains \bar{x}_{1j_ω} and $x_{2j_\omega+1}$, (3.16) can be canceled out from the proof of (i). And so is (3.17). As a result, we can get $\sum_{m=1}^n B_{12}^m(t) = 0$, and $\sum_{m=1}^n B_{kj}^m(t) = 0$. Therefore, according to (3.7) and (3.15), we obtain

$$\frac{d}{dt}W_{QDE}(t) = \frac{1}{2} \sum_{m=1}^n A_m(t) = \sum_i^n \Re a_{ii}(s) W_{QDE}(t) = \frac{1}{2} [\text{tr} A(s) + \text{tr} A^+(s)] W_{QDE}(t).$$

Integration above equation over $[t_0, t]$ follows the Liouville formula.

We need a lemma from (Theorem 2.10 [16]).

Lemma 3.11. *The necessary and sufficient condition of invertibility of quaternionic matrix $M(t_0)$ is $\text{ddet} M(t_0) \neq 0$ (or $W_{QDE}(t_0) \neq 0$).*

Proposition 3.12. *If $W_{QDE}(t) = 0$ at some t_0 in I then $x_1(t), x_2(t), \dots, x_n(t)$ are right dependent on I .*

Proof. From Liouville formula (Theorem 3.2), we have

$$W_{QDE}(t_0) = 0, \quad \text{implies} \quad W_{QDE}(t) = 0, \quad \text{for any } t \in I,$$

According to Lemma 3.11, the quaternionic matrix $M(t)$ is not invertible on I . Hence, the linear system

$$M(t)r = 0, \quad \text{or} \quad (x_1(t), x_2(t), \dots, x_n(t))r = 0, \quad r = (r_1, r_2, \dots, r_n)^\top \in \mathbb{H}^n,$$

has a non-zero solution. Consequently, the n solution $x_1(t), x_2(t), \dots, x_n(t)$ are dependent on I .

From Theorem 3.6, 3.8, 3.9, we immediately have

Theorem 3.13. *Let $A(t)$ in Eq.(3.1) be continuous functions of t on an interval I . There n solutions $x_1(t), x_2(t), \dots, x_n(t)$ of Eq.(3.1) on I are dependent on I if and only if the Wronskian $W_{QDE}(t)$ is zero at some t_0 in I .*

Now, we present two important results on the structure of the general solution.

Theorem 3.14. *There are n independent solutions $x_1(t), x_2(t), \dots, x_n(t)$ of Eq.(3.1).*

Proof. According to Theorem 3.1, we can easily choose n independent quaternionic vector $x_1^0, x_2^0, \dots, x_n^0$ in \mathbb{H}^n . (e.g. $x_1^0 = (1, 0, \dots, 0)^\top, x_2^0 = (0, 1, \dots, 0)^\top, \dots, x_n^0 = (0, 0, \dots, 1)^\top$). for any $k = 1, 2, \dots, n$ and $t_0 \in I$, Eq.(3.1) exists a unique solution $x_k(t)$ satisfying $x_k(t_0) = x_k^0$. If there are n quaternionic constants $r_1, r_2, \dots, r_n \in \mathbb{H}$ such that

$$x_1(t)r_1 + x_2(t)r_2 + \dots + x_n(t)r_n = 0, \quad \text{for any } t \in I.$$

In particularly, taking $t = t_0$, we have

$$x_1(t_0)r_1 + x_2(t_0)r_2 + \cdots + x_n(t_0)r_n = 0, \quad \text{i.e.,} \quad x_1^0 r_1 + x_2^0 r_2 + \cdots + x_n^0 r_n = 0.$$

Noting that $x_1^0, x_2^0, \dots, x_n^0$ is right independent, it follows that $r_1 = r_2 = \cdots = r_n = 0$. Thus, $x_1(t), x_2(t), \dots, x_n(t)$ of Eq.(3.1) are dependent on I .

Theorem 3.15. (*Structure of the general solution*) If $x_1(t), x_2(t), \dots, x_n(t)$ are n independent solutions of Eq.(3.1), then each solution of Eq.(3.1) is expressed as

$$x(t) = x_1(t)r_1 + x_2(t)r_2 + \cdots + x_n(t)r_n \quad (3.18)$$

r_1, r_2, \dots, r_n are undetermined quaternionic constants. The set of all the solutions is a free right-module.

Proof. For any $t_0 \in I$, let

$$x(t_0) = x_1(t_0)r_1 + x_2(t_0)r_2 + \cdots + x_n(t_0)r_n.$$

Since $x_1(t), x_2(t), \dots, x_n(t)$ are independent, by Theorem 3.13, getting $W_{QDE}(t_0) \neq 0$. Then exists a unique r_1, r_2, \dots, r_n (see Theorem 2.3 [21]). For determined r_1, r_2, \dots, r_n , from the superposition theorem, we know that $x_1(t)r_1 + x_2(t)r_2 + \cdots + x_n(t)r_n$ is also a solution of Eq.(3.1) satisfying the IVP $x(t_0)$. Therefore, $x(t)$ and $x_1(t)r_1 + x_2(t)r_2 + \cdots + x_n(t)r_n$ are two solutions of Eq.(3.1) satisfying the same IVP $x(t_0)$. By the uniqueness theorem (Theorem 3.1), the equality (3.18) holds.

4 Fundamental Matrix and Solution to QDEs

Definition 4.1. Let $x_1(t), x_2(t), \dots, x_n(t)$ be any n solutions of Eq.(3.1) on I . Then we call

$$M(t) = (x_1(t), x_2(t), \dots, x_n(t)) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}$$

as a solution matrix of Eq.(3.1). Moreover, if $x_1(t), x_2(t), \dots, x_n(t)$ be n independent solutions of Eq.(3.1) on I , the solution matrix of Eq.(3.1) is said to be a fundamental matrix of Eq.(3.1). Further, if $M(t_0) = E(\text{identity})$, $M(t)$ is said to be a normal fundamental matrix.

From Theorem 3.8, we know that if $M(t)$ is a fundamental matrix of Eq.(3.1), the Wronskian determinant $W_{QDE}(t) \neq 0$. By Theorem 3.15, we have

Theorem 4.2. Let $M(t)$ be a fundamental matrix of Eq.(3.1). Any solution $x(t)$ of Eq.(3.1) can be represented by

$$x(t) = M(t)q,$$

where q is a constant quaternionic vector. Moreover, for given IVP $x(t_0) = x^0$,

$$x(t) = M(t)M^{-1}(t_0)x^0.$$

By Theorem 3.13, we also have

Theorem 4.3. A solution matrix $M(t)$ of Eq.(3.1) on I is a fundamental matrix if and only if $\text{ddet}M(t) \neq 0$ (or $W_{QDE}(t) \neq 0$) on I . Moreover, for some t_0 in I such that $\text{ddet}M(t_0) \neq 0$ (or $W_{QDE}(t_0) \neq 0$), then $\text{ddet}M(t) \neq 0$ (or $W_{QDE}(t) \neq 0$).

Now consider the following quaternion-valued equations with constant quaternionic coefficients

$$\dot{x}(t) = Ax(t), \quad (4.1)$$

where $A \in \mathbb{H}^{n \times n}$ is a constant quaternion matrix. Then we have

Theorem 4.4. $M(t) = \exp\{At\}$ is a fundamental matrix of Eq.(4.1). Moreover, any solution $x(t)$ of Eq.(4.1) can be represented by

$$x(t) = \exp\{At\}q,$$

where $q \in \mathbb{H}^n$ is an arbitrary constant quaternion. For the IVP $x(t_0) = x^0$, any solution $x(t)$ of Eq.(4.1) can be represented by

$$x(t) = \exp\{A(t - t_0)\}x^0.$$

Now consider the diagonal homogenous system.

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} a_1(t) & 0 & \cdots & 0 \\ 0 & a_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad (4.2)$$

Theorem 4.5. Assume that the commutivity property

$$a_i(t) \int_{t_0}^t a_i(s)ds = \int_{t_0}^t a_i(s)ds a_i(t) \quad (4.3)$$

holds. The fundamental matrix can be chose by

$$M(t) = \begin{pmatrix} \exp\{\int_{t_0}^t a_1(s)ds\} & 0 & \cdots & 0 \\ 0 & \exp\{\int_{t_0}^t a_2(s)ds\} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp\{\int_{t_0}^t a_n(s)ds\} \end{pmatrix}.$$

Then the solution of the diagonal system Eq.(4.2) with the initial value $x(t_0) = x^0$ is given by

$$x(t) = M(t)x^0.$$

5 Algorithm for computing fundamental matrix

Since the fundamental matrix plays great role in solving QDEs, in this section, we provide two algorithms for computing fundamental matrix of linear QDEs with constant coefficients.

5.1 Method 1: using expansion of $\exp\{At\}$

From Theorem 4.5, we know that $\exp\{At\}$ is a fundamental matrix of Eq.(4.1). So if the coefficient matrix is not very complicate, we can use the definition of $\exp\{At\}$ to compute fundamental matrix of linear QDEs with constant coefficients.

Theorem 5.1. *If $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{H}^{n \times n}$ is a diagonal matrix, then*

$$\exp\{At\} = \begin{pmatrix} \exp\{\lambda_1 t\} & 0 & \cdots & 0 \\ 0 & \exp\{\lambda_2 t\} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp\{\lambda_n t\} \end{pmatrix}.$$

Proof. By the expansion,

$$\begin{aligned} \exp\{At\} &= E + \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \frac{t}{1!} + \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}^2 \frac{t^2}{2!} + \cdots \\ &= \begin{pmatrix} \exp\{\lambda_1 t\} & 0 & \cdots & 0 \\ 0 & \exp\{\lambda_2 t\} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp\{\lambda_n t\} \end{pmatrix}. \end{aligned}$$

If we can divide the matrix to some simple ones and use the expansion to compute the fundamental matrix.

$$A = \text{diag}A + N,$$

where N is a nilpotent matrix. That is, $N^n = 0$ and n is a finite number.

Example 5.1 Find a fundamental matrix of the following QDES

$$\dot{x} = Ax = \begin{pmatrix} \lambda & 1 & 0 \cdots 0 & 0 \\ 0 & \lambda & 1 \cdots 0 & 0 \\ & & \ddots & \ddots \\ 0 & 0 & 0 \cdots \lambda & 1 \\ 0 & 0 & 0 \cdots 0 & \lambda \end{pmatrix} x, \quad x = (x_1, x_2, \dots, x_k)^\top.$$

Answer. We see that $A = \lambda E + B$. Noticing that $(\lambda E)B = B(\lambda E)$, by Theorem 5.1, we have $\exp\{At\} = \exp\{\lambda Et\} \cdot \exp\{Bt\}$. where

$$B = \begin{pmatrix} 0 & 1 & 0 \cdots 0 & 0 \\ 0 & 0 & 1 \cdots 0 & 0 \\ & & \ddots & \ddots \\ 0 & 0 & 0 \cdots 0 & 1 \\ 0 & 0 & 0 \cdots 0 & 0 \end{pmatrix}$$

By B is a nilpotent matrix. That is, $B^k = 0$, we get

$$\exp\{Bt\} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} \cdots \frac{t^{k-2}}{(k-2)!} & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t \cdots \frac{t^{k-3}}{(k-3)!} & \frac{t^{k-2}}{(k-2)!} \\ & & \ddots & \ddots \\ 0 & 0 & 0 \cdots 1 & t \\ 0 & 0 & 0 \cdots 0 & 1 \end{pmatrix}$$

Then the fundamental matrix

$$\exp\{At\} = \exp\{\lambda Et\} \cdot \exp\{Bt\} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} \cdots \frac{t^{k-2}}{(k-2)!} & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t \cdots \frac{t^{k-3}}{(k-3)!} & \frac{t^{k-2}}{(k-2)!} \\ & & \ddots & \ddots \\ 0 & 0 & 0 \cdots 1 & t \\ 0 & 0 & 0 \cdots 0 & 1 \end{pmatrix} \exp\{\lambda t\}$$

We remark that this method can not be applied extensively when the coefficient matrix A is complicate. Sometimes, the two divisions of A can not be communicate. For example,

$$\begin{pmatrix} \mathbf{i} & 1 \\ 0 & \mathbf{j} \end{pmatrix} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In view of $\mathbf{i} \neq \mathbf{j}$, we see that

$$\begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{pmatrix}$$

In this case, this method can not be used. So we should find more effective method to compute the fundamental matrix. In what follows, we try to use the eigenvalue and eigenvector theory to compute it.

5.2 Method 2: eigenvalue and eigenvector theory

A quaternion λ is said to be a right eigenvalue of A if

$$Ax = x\lambda$$

for some nonzero (column) vector x with quaternion components. Similarly, a quaternion λ is a left eigenvalue of A if

$$Ax = \lambda x$$

for some nonzero (column) vector x with quaternion components. Right and left eigenvalues are in general unrelated. Usually, right and left eigenvalues are different.

In this paper, we emphasize on finding the solution taking the form

$$x = qe^{\lambda t}, \quad q = (q_1, q_2, \dots, q_n)^\top,$$

where λ is a quaternionic constant and q is a constant quaternionic vector. Substituting it into Eq.(4.1), we have

$$q\lambda e^{\lambda t} = Aqe^{\lambda t}.$$

Because $e^{\lambda t} \neq 0$, it follows that

$$q\lambda = Aq,$$

or

$$Aq = q\lambda. \tag{5.1}$$

So if we find such eigenvalue λ and eigenvector q in Eq.(5.1), we will find the solution of Eq.(4.1). We also say that the eigenvalue λ and eigenvector q are the right eigenvalue and eigenvector of the QDEs in Eq.(4.1).

Notice also that if $0 \neq \alpha \in \mathbb{H}$, then

$$Aq = q\lambda \Rightarrow Aq\alpha = q(\alpha^{-1}\lambda\alpha),$$

so we can sensibly talk about the eigenline spanned by an eigenvector q , even though there may be many associated eigenvalues! In what follows, if

$$\theta = \alpha^{-1}\lambda\alpha$$

we call it that θ is similar to λ . If there exists a nonsingular matrix T such that

$$A = T^{-1}BT,$$

we call it that A is similar to B , denoted by $A \sim B$.

Remark 5.2. If λ is a characteristic root of A , then so is $\alpha^{-1}\lambda\alpha$.

Remark 5.3. If θ, λ are two characteristic roots of A and θ is similar to λ , for any the eigenvector q of θ , there exists an eigenvector q' of λ such that q, q' are dependent.

From the definition of fundamental matrix, we have

Theorem 5.4. If the matrix A has n independent eigenvectors q_1, q_2, \dots, q_n , corresponding to the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (λ_i and λ_j can be similar), then

$$M(t) = (q_1 e^{\lambda_1 t}, q_2 e^{\lambda_2 t}, \dots, q_n e^{\lambda_n t})$$

is a fundamental matrix of Eq.(4.1).

Proof. From above discussion, we know $q_1 e^{\lambda_1 t}, q_2 e^{\lambda_2 t}, \dots, q_n e^{\lambda_n t}$ are n solution of Eq.(4.1). Thus, $M(t)$ is a solution matrix of Eq.(4.1). Moreover, by using the independence of q_1, q_2, \dots, q_n , we have

$$\text{ddet} M(0) = \text{ddet}(q_1, q_2, \dots, q_n) \neq 0.$$

Therefore, $M(t)$ is a fundamental matrix of Eq.(4.1).

Now we need a lemma from [3] (Proposition 2.4).

Lemma 5.5. *Suppose that $\lambda_1, \lambda_2, \dots, \lambda_r$ are distinct eigenvalues for A , no two of which are similar, and let q_1, \dots, q_r be corresponding eigenvectors. Then q_1, \dots, q_r are linearly independent.*

Then by Lemma 5.5, we have the corollary

Corollary 5.6. *If the matrix A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, no two of which are conjugate, then*

$$M(t) = (q_1 e^{\lambda_1 t}, q_2 e^{\lambda_2 t}, \dots, q_n e^{\lambda_n t})$$

is a fundamental matrix of Eq.(4.1).

Now we need some results from [5] (Theorem 11, Theorem 12).

Lemma 5.7. *If A is in triangular form, then every diagonal element is a characteristic root.*

Lemma 5.8. *Let a matrix of quaternion be in triangular form. Then the only characteristic roots are the diagonal elements (and the numbers similar to them).*

Lemma 5.9. *Similar matrices have the same characteristic roots.*

Lemma 5.10. [5] (Theorem 2) *Every matrix of quaternion can be transformed into triangular form by a unitary matrix.*

Example 5.2 Find a fundamental matrix of the following QDES

$$\dot{x} = \begin{pmatrix} \mathbf{i} & 1 \\ 0 & 1 + \mathbf{i} \end{pmatrix} x, \quad x = (x_1, x_2)^\top. \quad (5.2)$$

Answer: From Lemma 5.7 and Lemma 5.8, we see that $\lambda_1 = \mathbf{i}$ and $\lambda_2 = 1 + \mathbf{i}$. To find the eigenvector of $\lambda_1 = \mathbf{i}$, we consider the following equation

$$Aq = q\lambda_1,$$

that is

$$\begin{cases} \mathbf{i}q_1 + q_2 &= q_1 \mathbf{i} \\ (1 + \mathbf{i})q_2 &= q_2 \mathbf{i} \end{cases} \quad (5.3)$$

From the second equation of (5.3), if we take $q_2 = 0$. Substituting it into the first equation of (5.3), we can take $q_1 = 1$. So we obtain one eigenvector as

$$\nu_1 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To find the eigenvector of $\lambda_2 = 1 + i$, we consider the following equation

$$Aq = q\lambda_2,$$

that is

$$\begin{cases} iq_1 + q_2 &= q_1(1 + i) \\ (1 + i)q_2 &= q_2(1 + i) \end{cases} \quad (5.4)$$

From the second equation of (5.4), if we take $q_2 = 1$. Substituting it into the first equation of (5.4), we can take $q_1 = 1$. So we get another eigenvector as

$$\nu_2 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

By Corollary 5.6

$$M(t) = (\nu_1 e^{\lambda_1 t}, \nu_2 e^{\lambda_2 t}) = \begin{pmatrix} e^{it} & e^{(1+i)t} \\ 0 & e^{(1+i)t} \end{pmatrix},$$

is a fundamental matrix of Eq.(5.2). In fact, by the definition of fundamental matrix, we can also verify that $M(t)$ is a fundamental matrix of Eq.(5.2). Now we verify the fundamental matrix as follows. First, we show that $M(t)$ is a solution matrix of Eq.(5.2). Let $\phi_1(t) = \nu_1 e^{\lambda_1 t}$ and $\phi_2(t) = \nu_2 e^{\lambda_2 t}$, then

$$\dot{\phi}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} i e^{it} = \begin{pmatrix} i & 1 \\ 0 & 1 + i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{it} = \begin{pmatrix} i & 1 \\ 0 & 1 + i \end{pmatrix} \phi_1(t),$$

which implies that $\phi_1(t)$ is a solution of Eq.(5.2). Similarly,

$$\dot{\phi}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 + i) e^{(1+i)t} = \begin{pmatrix} i & 1 \\ 0 & 1 + i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} i & 1 \\ 0 & 1 + i \end{pmatrix} \phi_2(t),$$

which implies that $\phi_2(t)$ is another solution of Eq.(5.2). Therefore, $M(t) = (\phi_1(t), \phi_2(t))^\top$ is a solution matrix of Eq.(5.2).

Secondly, by Theorem 4.3, and taking $t_0 = 0$,

$$\text{ddet}W(t_0) = \text{ddet}W(0) = \text{ddet}(\nu_1, \nu_2) = \text{ddet} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \neq 0.$$

then $\text{ddet}M(t) \neq 0$. Therefore, $M(t)$ is a fundamental matrix of Eq.(5.2).

6 System with multiple eigenvalues

In this section, we will give an algorithm to construct fundamental matrix when system have the multiple eigenvalues. There are two cases. One case is that the numbers of eigenvectors are equal to the dimension of the system. The other case is that the numbers of eigenvectors less than the dimension of the system (that is to say, not enough eigenvectors).

6.1 Multiple eigenvalues with enough eigenvectors

In this subsection, we give two examples. Example 2 has two identity eigenvalues. Example 3 has two eigenvalues which is similar. Two similar eigenvalues can be seen as two multiple eigenvalues.

Example 6.1 Find a fundamental matrix of the following QDEs

$$\dot{x} = \begin{pmatrix} j & i \\ 0 & j \end{pmatrix} x, \quad x = (x_1, x_2)^\top. \quad (6.1)$$

Answer: From Lemma 5.7 and Lemma 5.8, we see that $\lambda_{1,2} = j$. To find the eigenvector of $\lambda_{1,2} = j$, we consider the following equation

$$Aq = q\lambda_{1,2},$$

that is

$$\begin{cases} jq_1 + iq_2 = q_1j \\ jq_2 = q_2j \end{cases} \quad (6.2)$$

From the second equation of (6.16), if we take $q_2 = 0$. Substituting it into the first equation of (6.16), we can take $q_1 = 1$. So we obtain one eigenvector as

$$\nu_1 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

If we take $q_2 = 1$. Substituting it into the first equation of (6.16), we can take $q_1 = -\frac{k}{2}$. So we get another eigenvector as

$$\nu_2 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -\frac{k}{2} \\ 1 \end{pmatrix}$$

Since

$$\text{ddet}(\nu_1, \nu_2) = \text{ddet} \begin{pmatrix} 1 & -\frac{k}{2} \\ 0 & 1 \end{pmatrix} = \det \left[\begin{pmatrix} 1 & 0 \\ \frac{k}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{k}{2} \\ 0 & 1 \end{pmatrix} \right] = 1 \neq 0,$$

the eigenvectors ν_1 and ν_2 are independent. Taking

$$M(t) = (\nu_1 e^{jt}, \nu_2 e^{jt}) = \begin{pmatrix} e^{jt} & -\frac{k}{2} e^{jt} \\ 0 & e^{jt} \end{pmatrix},$$

From Theorem 5.4, $M(t)$ is a fundamental matrix. In fact, by the definition of fundamental matrix, we can verify that $M(t)$ is a fundamental matrix of Eq.(6.1). Now we verify the fundamental matrix as follows. First, we show that $M(t)$ is a solution matrix of Eq.(6.1). Let $\phi_1(t) = \nu_1 e^{\lambda_{1,2}t}$ and $\phi_2(t) = \nu_2 e^{\lambda_{1,2}t}$, then

$$\dot{\phi}_1(t) = \begin{pmatrix} \mathbf{j}e^{\mathbf{j}t} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{j} & \mathbf{i} \\ 0 & \mathbf{j} \end{pmatrix} \begin{pmatrix} e^{\mathbf{j}t} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{j} & \mathbf{i} \\ 0 & \mathbf{j} \end{pmatrix} \phi_1(t),$$

which implies that $\phi_1(t)$ is a solution of Eq.(6.1). Similarly, then

$$\dot{\phi}_2(t) = \begin{pmatrix} -\frac{\mathbf{k}}{2}\mathbf{j}e^{\mathbf{j}t} \\ \mathbf{j}e^{\mathbf{j}t} \end{pmatrix} = \begin{pmatrix} \mathbf{j} & \mathbf{i} \\ 0 & \mathbf{j} \end{pmatrix} \begin{pmatrix} -\frac{\mathbf{k}}{2} \\ 1 \end{pmatrix} e^{\mathbf{j}t} = \begin{pmatrix} \mathbf{j} & \mathbf{i} \\ 0 & \mathbf{j} \end{pmatrix} \phi_2(t),$$

which implies that $\phi_2(t)$ is another solution of Eq.(6.1). Therefore, $M(t) = (\phi_1(t), \phi_2(t))^\top$ is a solution matrix of Eq.(6.1).

Secondly, by Theorem 4.3, and taking $t_0 = 0$, Since

$$\text{ddet}M(t_0) = \text{ddet}M(0) = \text{ddet}(\nu_1, \nu_2) = \text{ddet} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \det \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \neq 0,$$

then $\text{ddet}M(t) \neq 0$. Therefore, $M(t)$ is a fundamental matrix of Eq. (6.1).

6.2 Multiple eigenvalues with fewer eigenvectors

For any $A \in \mathbb{H}^{n \times n}$, if we obtain double or multiple eigenvalues. This means that the number of independent eigenvectors might be smaller than the dimensionality of the system. So we may not get a fundamental matrix. We therefore have to discover how to find the “missing solutions”. In this case, first, we need to prove the following basic results.

Now we need a lemma from [30] (Theorem 5.4).

Lemma 6.1. *Any $n \times n$ quaternion matrix A has exactly n (right) eigenvalues which are complex numbers with nonnegative imaginary parts.*

Remark 6.2. *For the convenience of studying, we can just consider the eigenvalues which are complex numbers with nonnegative imaginary parts. And those eigenvalues are said to be the standard eigenvalues of A (Obviously, no two of which are similar).*

Let $A \in \mathbb{H}^{n \times n}$, $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct standard eigenvalues for A , the multiplicity of all the eigenvalues are n_1, n_2, \dots, n_k respectively, and $n_1 + n_2 + \dots + n_k = n$. For the independent eigenvectors $v_1^{j_1}, v_1^{j_2}, \dots, v_1^{j_{r_j}}$ associated with eigenvalue λ_j of multiplicity n_j ($r_j \leq n_j$). A set $\{v_1^{j_i}, v_2^{j_i}, \dots, v_{m_i}^{j_i}\}$ ($m_i \leq n_j$) based on the eigenvector $v_1^{j_i}$ ($Av_1^{j_i} = \lambda_j v_1^{j_i}$ and $i \in \{1, 2, \dots, r_j\}$)

such that

$$\begin{aligned}
Av_{m_{ji}}^{ji} - v_{m_{ji}}^{ji} \lambda_j &= v_{m_{ji}-1}^{ji} \\
Av_{m_{ji}-1}^{ji} - v_{m_{ji}-1}^{ji} \lambda_j &= v_{m_{ji}-2}^{ji} \\
&\vdots \\
Av_2^{ji} - v_2^{ji} \lambda_j &= v_1^{ji}
\end{aligned} \tag{6.3}$$

Note that $N_{\lambda_j}^{m_{ji}} = \{\sum_{l=1}^{m_{ji}} v_l^{ji} r_l | r_l \in \mathbb{H}\}$, And $N_{\lambda_j}^{m_{ji}} \subseteq \mathbb{H}^n$ is submodule.

Now we can get the following result according to Lemma 3.7, Theorem 2 in [29] and some basic theories of direct sum.

Theorem 6.3. *For all submodules $N_{\lambda_j}^{m_{ji}}$, $i \in \{1, 2, \dots, r_j\}$ and $j \in \{1, 2, \dots, k\}$, then there exists the following decomposition*

$$\mathbb{H}^n = N_{\lambda_1}^{m_{11}} \oplus N_{\lambda_1}^{m_{12}} \oplus \dots \oplus N_{\lambda_1}^{m_{1r_1}} \oplus N_{\lambda_2}^{m_{21}} \oplus \dots \oplus N_{\lambda_2}^{m_{2r_2}} \oplus \dots \oplus N_{\lambda_k}^{m_{k1}} \oplus \dots \oplus N_{\lambda_k}^{m_{kr_k}} \tag{6.4}$$

where $N_{\lambda_j}^{m_{ji}} = \{\sum_{l=1}^{m_{ji}} v_l^{ji} r_l | r_l \in \mathbb{H}\}$.

That's to say, for any $u \in \mathbb{H}^n$, there exist the unique vectors $u_1^{m_{11}}, \dots, u_1^{m_{1r_1}}, u_2^{m_{21}}, \dots, u_2^{m_{2r_2}}, \dots, u_k^{m_{k1}}, \dots, u_k^{m_{kr_k}}$, where $u_j^{m_{ji}} \in N_{\lambda_j}^{m_{ji}}$, such that

$$u = u_1^{m_{11}} + \dots + u_1^{m_{1r_1}} + u_2^{m_{21}} + \dots + u_2^{m_{2r_2}} + \dots + u_k^{m_{k1}} + \dots + u_k^{m_{kr_k}} = \sum_{j=1}^k \sum_{i=1}^{r_j} u_j^{m_{ji}}. \tag{6.5}$$

By Eq.(6.5) and $N_{\lambda_j}^{m_{ji}}$, any solution $x(t) = \exp\{At\}\eta$ of Eq.(4.1) can be represented by

$$\begin{aligned}
x(t) &= (\exp At)\eta = (\exp At) \sum_{j=1}^k \sum_{i=1}^{r_j} u_j^{m_{ji}} = \sum_{j=1}^k \sum_{i=1}^{r_j} (\exp At) u_j^{m_{ji}} \\
&= \sum_{j=1}^k \sum_{i=1}^{r_j} (\exp At) (v_1^{ji} r_1 + v_2^{ji} r_2 + \dots + v_{m_{ji}}^{ji} r_{m_{ji}}) \\
&= \sum_{j=1}^k \sum_{i=1}^{r_j} \sum_{s=0}^{+\infty} (At)^s (v_1^{ji} r_1 + v_2^{ji} r_2 + \dots + v_{m_{ji}}^{ji} r_{m_{ji}}) \\
&= \sum_{j=1}^k \sum_{i=1}^{r_j} \sum_{s=0}^{+\infty} (At)^s \sum_{l=1}^{m_{ji}} v_l^{ji} r_l \\
&= \sum_{j=1}^k \sum_{i=1}^{r_j} \sum_{l=1}^{m_{ji}} \sum_{s=0}^{+\infty} (At)^s v_l^{ji} r_l
\end{aligned} \tag{6.6}$$

where $\eta \in \mathbb{H}^n$. According to Eq.(6.3), then

$$\begin{aligned}
(At)^0 v_l^{ji} r_l &= v_l^{ji} r_l \\
(At)^1 v_l^{ji} r_l &= t(v_{l-1}^{ji} + v_l^{ji} \lambda_j) r_l \\
(At)^2 v_l^{ji} r_l &= \frac{t^2}{2!} (v_{l-2}^{ji} + 2v_{l-1}^{ji} \lambda_j + v_l^{ji} \lambda_j^2) r_l \\
(At)^3 v_l^{ji} r_l &= \frac{t^3}{3!} (C_3^3 v_{l-3}^{ji} + C_3^2 v_{l-2}^{ji} \lambda_j + C_3^1 v_{l-1}^{ji} \lambda_j^2 + v_l^{ji} \lambda_j^3) r_l \\
&\vdots \\
(At)^{l-1} v_l^{ji} r_l &= \frac{t^{l-1}}{(l-1)!} (C_{l-1}^{l-1} v_1^{ji} + C_{l-1}^{l-2} v_2^{ji} \lambda_j + \cdots + C_{l-1}^1 v_{l-1}^{ji} \lambda_j^{l-2} + v_p^{ji} \lambda_j^{l-1}) r_l \\
(At)^l v_l^{ji} r_l &= \frac{t^l}{(l)!} (C_l^{l-1} v_1^{ji} \lambda_j + C_l^{l-2} v_2^{ji} \lambda_j^2 + \cdots + C_l^1 v_{l-1}^{ji} \lambda_j^{l-1} + v_p^{ji} \lambda_j^l) r_l \\
(At)^{l+1} v_l^{ji} r_l &= \frac{t^{l+1}}{(l+1)!} (C_{l+1}^{l-1} v_1^{ji} \lambda_j^2 + C_{l+1}^{l-2} v_2^{ji} \lambda_j^3 + \cdots + C_{l+1}^1 v_{l-1}^{ji} \lambda_j^l + v_p^{ji} \lambda_j^{l+1}) r_l \\
&\vdots
\end{aligned} \tag{6.7}$$

Substituting Eq.(6.7) into $\sum_{s=0}^{+\infty} (At)^s v_l^{ji} r_l$, we can get

$$\sum_{s=0}^{+\infty} (At)^s v_l^{ji} r_l = (v_l^{ji} + t v_{l-1}^{ji} + \frac{t^2}{2!} v_{l-2}^{ji} + \cdots + \frac{t^{l-2}}{(l-2)!} v_2^{ji} + \frac{t^{l-1}}{(l-1)!} v_1^{ji}) (\exp \lambda_j t) r_l \tag{6.8}$$

Consequently, we have

$$x(t) = (\exp At) \eta = \sum_{j=1}^k \sum_{i=1}^{r_j} \sum_{l=1}^{m_{ji}} (v_l^{ji} + t v_{l-1}^{ji} + \frac{t^2}{2!} v_{l-2}^{ji} + \cdots + \frac{t^{l-1}}{(l-1)!} v_1^{ji}) (\exp \lambda_j t) r_l \tag{6.9}$$

Therefore, if the A is real or complex matrix, the form of solution $x(t) = \exp\{At\}\eta$ of Eq.(4.1) is the same as that ordinary form.

Secondly, how to get the solution $x(t) = \exp\{At\}\eta$ of Eq.(6.9)? If we get the eigenvalue λ_j , By $Av_1^{ji} = v_1^{ji} \lambda_j$ and Eq.(6.3), we can get the set $\{v_{1_i}^j, v_{2_i}^j, \cdots, v_{m_{ji}-1}^j, v_{m_{ji}}^j\}$. For computational convenience, we introduce a method for computing the eigenvalue λ_j and the set $\{v_{1_i}^j, v_{2_i}^j, \cdots, v_{m_{ji}-1}^j, v_{m_{ji}}^j\}$. First, we will introduce the following results.

For $A \in \mathbb{H}^{n \times n}$, note $A = A_1 + A_2 \mathbf{j}$, where A_1 and A_2 are $n \times n$ complex matrices. We associate with A the $2n \times 2n$ complex matrix

$$\phi(A) = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} \tag{6.10}$$

and call $\phi(A)$ the complex adjoint matrix of the quaternion matrix A . Let Σ be the collection of all $2n \times 2n$ partitioned complex matrices in the form (6.10).

For $v \in \mathbb{H}^n$, write $v = v_1 + v_2 \mathbf{j}$, where v_1 and v_2 are complex n -tuples. We associate with v the complex $2n$ -tuples.

$$\varphi(v) = \begin{pmatrix} v_1 \\ -\overline{v_2} \end{pmatrix} \tag{6.11}$$

The mapping $v \rightarrow \varphi(v)$ is an isomorphism between \mathbb{H}^n and \mathbb{C}^{2n} obviously. And $v \neq 0$ if and only if $\varphi(v) \neq 0$ can be easily obtained. And $\varphi(v)^*$ is called the adjoint vector of the $\varphi(v)$

$$\varphi(v)^* = \begin{pmatrix} v_1 \\ -\overline{v_2} \end{pmatrix}^* = \begin{pmatrix} v_2 \\ \overline{v_1} \end{pmatrix}$$

Lemma 6.4. For $A \in \mathbb{H}^{n \times n}$, $v, u \in \mathbb{H}^n$ and $\lambda \in \mathbb{C}$, if $\phi(A)\varphi(v) = \varphi(u) + \varphi(v)\lambda$ holds, then $\phi(A)\varphi(v)^* = \varphi(u)^* + \varphi(v)^*\overline{\lambda}$ holds

It can be easily proved by Lemma 3 [29].

Lemma 6.5. For $A \in \mathbb{H}^{n \times n}$, $v, u \in \mathbb{H}^n$ and $\lambda \in \mathbb{C}$, if $Av = u + v\lambda$ holds, if and only if $\phi(A)\varphi(v) = \varphi(u) + \varphi(v)\lambda$ holds.

Proof. since

$$Av = (A_1 + A_2\mathbf{j})(v_1 + v_2\mathbf{j}) = A_1v_1 + A_2\mathbf{j}v_2\mathbf{j} + A_2\mathbf{j}v_1 + A_1v_2\mathbf{j}$$

$$u + v\lambda = (u_1 + u_2\mathbf{j}) + (v_1 + v_2\mathbf{j})\lambda = u_1 + v_1\lambda + (u_2\mathbf{j} + v_2\mathbf{j}\lambda)$$

if $Av = u + v\lambda$ holds, then

$$\begin{aligned} A_1v_1 + A_2\mathbf{j}v_2\mathbf{j} &= u_1 + v_1\lambda \\ A_2\mathbf{j}v_1 + A_1v_2\mathbf{j} &= u_2\mathbf{j} + v_2\mathbf{j}\lambda \end{aligned} \tag{6.12}$$

It implies

$$\begin{aligned} \frac{A_1v_1 - A_2\overline{v_2}}{A_2v_1 + A_1\overline{v_2}} &= \frac{u_1 + v_1\lambda}{\overline{u_2} + \overline{v_2}\lambda} \end{aligned} \tag{6.13}$$

It follows that

$$\begin{pmatrix} \frac{A_1}{-A_2} & \frac{A_2}{A_1} \end{pmatrix} \begin{pmatrix} v_1 \\ -\overline{v_2} \end{pmatrix} = \begin{pmatrix} u_1 \\ -\overline{u_2} \end{pmatrix} + \begin{pmatrix} v_1 \\ -\overline{v_2} \end{pmatrix} \lambda \tag{6.14}$$

Therefore

$$\phi(A)\varphi(v) = \varphi(u) + \varphi(v)\lambda$$

Conversely, it can be easily proved.

Corollary 6.6. For $A \in \mathbb{H}^{n \times n}$, $v \in \mathbb{H}^n$ and $\lambda \in \mathbb{C}$, if $Av = v\lambda$ holds, if and only if $\phi(A)\varphi(v) = \varphi(v)\lambda$ holds.

Secondly, to get λ_j and the set $\{v_{1_i}^j, v_{2_i}^j, \dots, v_{m_{ji}-1}^j, v_{m_{ji}}^j\}$, we introduce the computational process of this method according to the proof of Theorem 1 [29].

Let $\lambda_j = a + b\mathbf{i}$ (k-fold) is a eigenvalue of $\phi(A)$, $A \in \mathbb{H}^{n \times n}$. By Corollary 6.6 and theorem 1 [29]. Then

- (i) If $b > 0$, $\lambda_j = a + b\mathbf{i}$ (k -fold) is a eigenvalue of A and set $\{\varphi(v_{1_i}^j), \varphi(v_{2_i}^j), \dots, \varphi(v_{m_{ji}-1}^j), \varphi(v_{m_{ji}}^j)\}$ is easily calculated. (the set $\{v_{1_i}^j, v_{2_i}^j, \dots, v_{m_{ji}-1}^j, v_{m_{ji}}^j\}$ is undetermined). By Eq.(6.11), we can obtain the set $\{v_{1_i}^j, v_{2_i}^j, \dots, v_{m_{ji}-1}^j, v_{m_{ji}}^j\}$.
- (ii) If $b = 0$, $\lambda_j = a + b\mathbf{i}$ ($\frac{k}{2}$ -fold) is a eigenvalue of A , and by Lemma 6.1, there two sets $\{\varphi(v_{1_i}^j), \varphi(v_{2_i}^j), \dots, \varphi(v_{m_{ji}-1}^j), \varphi(v_{m_{ji}}^j)\}$ and $\{\varphi(v_{1_i}^j)^*, \varphi(v_{2_i}^j)^*, \dots, \varphi(v_{m_{ji}-1}^j)^*, \varphi(v_{m_{ji}}^j)^*\}$ are calculated. Taking the set $\{\varphi(v_{1_i}^j), \varphi(v_{2_i}^j), \dots, \varphi(v_{m_{ji}-1}^j), \varphi(v_{m_{ji}}^j)\}$, By Eq.(6.11), we can obtain the set $\{v_{1_i}^j, v_{2_i}^j, \dots, v_{m_{ji}-1}^j, v_{m_{ji}}^j\}$.

Finally, to obtain $\exp\{At\}$ from Eq.(6.9), we can firstly choose n independent initial value vector, then the corresponding n solutions to the IVP are independent. For convenience, we usually choose the natural basis. Let $\eta = e_1, \eta = e_2, \dots, \eta = e_n$, correspondingly, we can get n independent solutions. These n independent solutions compose the column of $\exp\{At\}$. Noticing that $\exp\{At\} = \exp\{At\}E = [(\exp\{At\})e_1, (\exp\{At\})e_2, \dots, (\exp\{At\})e_n]$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

are the unit vector.

Some examples are presented to show the validity of this method.

Example 6.2 Find a fundamental matrix of the following QDEs

$$\dot{x} = \begin{pmatrix} \mathbf{i} & 1 \\ 0 & \mathbf{j} \end{pmatrix} x, \quad x = (x_1, x_2)^T. \quad (6.15)$$

Answer: From Lemma 5.7 and Lemma 5.8, we see that $\lambda_1 = \mathbf{i}$ and $\lambda_2 = \mathbf{j}$. It should be noted that \mathbf{j} is similar to \mathbf{i} . In fact, taking $\alpha = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$, then $\alpha^{-1} = \frac{1}{4}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k})$. Consequently, $\mathbf{j} = \alpha^{-1}\mathbf{i}\alpha$, that is, \mathbf{j} is similar to \mathbf{i} . To find the eigenvector of $\lambda_1 = i$, we consider the following equation

$$Aq = q\lambda_1,$$

that is

$$\begin{cases} \mathbf{i}q_1 + q_2 &= q_1\mathbf{i} \\ \mathbf{j}q_2 &= q_2\mathbf{i} \end{cases} \quad (6.16)$$

Let $q_1 = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $q_2 = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Substituting it into the Eq.(6.16), we obtain $q_1 = a_0 + a_1\mathbf{i}$, $q_2 = 0$, $a_0, a_1 \in \mathbb{R}$.

According to Remark 5.3, it is impossible to find two independent eigenvector of (6.15).

We consider matrix $\phi(A)$, the eigenvalue of $\phi(A)$ are $\lambda_1 = \mathbf{i}$ (2-fold), $\lambda_2 = -\mathbf{i}$ (2-fold). The eigenvector of $\lambda_1 = \mathbf{i}$ is $\varphi(v) = (1, 0, 0, 0)^T$, and

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

By Lemma 6.5 and $\phi(A)\varphi(u) = \varphi(v) + \varphi(u)\lambda_1$, we can get $\varphi(u) = (0, 1, \frac{1}{2}, \mathbf{i})^T$ and

$$u = \begin{pmatrix} -\frac{1}{2}\mathbf{j} \\ 1 + \mathbf{k} \end{pmatrix}$$

Substituting v, u into Eq.(6.9), For any solutions $\exp\{At\}\eta$, let $\eta = vr_1 + ur_2$ we can get

$$\exp\{At\}\eta = ve^{it}r_1 + (u + vt)e^{it}r_2$$

Namely

$$\exp\{At\}\eta = \begin{pmatrix} 1 & -\frac{1}{2}\mathbf{j} + t \\ 0 & 1 + \mathbf{k} \end{pmatrix} \begin{pmatrix} e^{it}r_1 \\ e^{it}r_2 \end{pmatrix}$$

Let $\eta = (1, 0)^T, (0, 1)^T$ in turn, we can obtain two linear independent solutions, which compose the fundamental matrix $\exp\{At\}$, namely

$$\exp\{At\} = \begin{pmatrix} e^{it} & e^{it}\frac{1-\mathbf{k}}{2} - (\frac{1}{2}\mathbf{j} - t)e^{it}\frac{-\mathbf{i}+\mathbf{j}}{4} \\ 0 & (1 + \mathbf{k})e^{it}\frac{-\mathbf{i}+\mathbf{j}}{4} \end{pmatrix}$$

Example 6.3 Find fundamental matrix $\exp\{At\}$ of the following QDES

$$\dot{x} = Ax = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ \mathbf{k} & 1 & \mathbf{k} \\ 0 & 0 & 1 \end{pmatrix} x, \quad x = (x_1, x_2, x_3)^T. \quad (6.17)$$

Answer: we can easily get the eigenvalue of $\phi(A)$ are $\lambda_1 = 0$ (2-fold), $\lambda_2 = 1 + \mathbf{i}$, $\lambda_3 = 1 - \mathbf{i}$ and $\lambda_4 = 1$ (2-fold). The eigenvector of $\lambda_1 = 0$ is $\varphi(v_1) = (0, -\mathbf{i}, 0, 1, 0, 0)^T$. By Eq.(6.11), we get

$$v_1 = \begin{pmatrix} -\mathbf{j} \\ -\mathbf{i} \\ 0 \end{pmatrix}.$$

The eigenvectors of $\lambda_2 = 1 + \mathbf{i}$ are $\varphi(v_2) = (1, 0, 0, 0, 1, 0)^T$ and

$$v_2 = \begin{pmatrix} 1 \\ -\mathbf{j} \\ 0 \end{pmatrix}.$$

The eigenvectors of $\lambda_4 = 1$ are $\varphi(v_3) = (\frac{1+\mathbf{i}}{2}, \frac{1+\mathbf{i}}{2}, -\frac{1+\mathbf{i}}{2}, 0, 1, 0)^T$ and

$$v_3 = \begin{pmatrix} \frac{1+\mathbf{i}}{2} \\ \frac{1+\mathbf{i}}{2} - \mathbf{j} \\ -\frac{1+\mathbf{i}}{2} \end{pmatrix}.$$

From corollary 5.6, then

$$M(t) = \begin{pmatrix} -\mathbf{j} & e^{(1+\mathbf{i})t} & \frac{1+\mathbf{i}}{2}e^t \\ -\mathbf{i} & -\mathbf{j}e^{(1+\mathbf{i})t} & (\frac{1+\mathbf{i}}{2}-\mathbf{j})e^t \\ 0 & 0 & -\frac{1+\mathbf{i}}{2}e^t \end{pmatrix}.$$

is a fundamental matrix of Eq.(6.17).

For any solutions $\exp\{At\}\eta$ of Eq.(6.17). Substituting v_1, v_2, v_3 into Eq.(6.9), let $\eta = v_1r_1 + v_2r_2 + v_3r_3$ we can get

$$\exp\{At\}\eta = v_1r_1 + v_2e^{(1+\mathbf{i})t}r_2 + v_3e^tr_3,$$

namely

$$\exp\{At\}\eta = \begin{pmatrix} -\mathbf{j} & 1 & \frac{1+\mathbf{i}}{2} \\ -\mathbf{i} & -\mathbf{j} & \frac{1+\mathbf{i}}{2}-\mathbf{j} \\ 0 & 0 & -\frac{1+\mathbf{i}}{2} \end{pmatrix} \begin{pmatrix} r_1 \\ e^{(1+\mathbf{i})t}r_2 \\ e^tr_3 \end{pmatrix}.$$

Let $\eta = (1, 0, 0)^T, (0, 1, 0)^T, (0, 1, 0)^T$ in turn, we can obtain three linear independent solutions, which compose the fundamental matrix $\exp\{At\}$, namely

$$\exp\{At\} = \begin{pmatrix} \frac{1-\mathbf{i}}{2} + \frac{1+\mathbf{i}}{2}\alpha & \frac{-\mathbf{j}+\mathbf{k}}{2} + \beta & \mathbf{j}\gamma + \delta - e^t \\ \frac{\mathbf{j}-\mathbf{k}}{2} - \frac{\mathbf{j}-\mathbf{k}}{2}\alpha & \frac{1-\mathbf{i}}{2} - \mathbf{j}\beta & \mathbf{i}\gamma - \mathbf{j}\delta - (1-\mathbf{j}-\mathbf{k})e^t \\ 0 & 0 & e^t \end{pmatrix}.$$

where $\alpha = e^{(1+\mathbf{i})t}$, $\beta = e^{(1+\mathbf{i})t}\frac{\mathbf{j}-\mathbf{k}}{2}$, $\gamma = \frac{1+\mathbf{i}+\mathbf{j}-\mathbf{k}}{2}$, $\delta = e^{(1+\mathbf{i})t}\frac{1-\mathbf{i}+\mathbf{j}-\mathbf{k}}{2}$.

Example 6.4 Find fundamental matrix $\exp\{At\}$ of the following QDEs

$$\dot{x} = Ax = \begin{pmatrix} \mathbf{j} & \mathbf{k} & \mathbf{i} \\ 0 & 1 & \mathbf{k} \\ 0 & 0 & 1 \end{pmatrix} x, \quad x = (x_1, x_2, x_3)^T. \quad (6.18)$$

Answer: we easily get the eigenvalue of $\phi(A)$ are $\lambda_1 = \mathbf{i}$, $\lambda_2 = -\mathbf{i}$ and $\lambda_3 = 1$ (4-fold). The eigenvector of $\lambda_1 = \mathbf{i}$ is $\varphi(v_1) = (-\mathbf{i}, 0, 0, 1, 0, 0)^T$, and

$$v_1 = \begin{pmatrix} -\mathbf{i} - \mathbf{j} \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvectors of $\lambda_3 = 1$ are $\varphi(v_2) = (-\mathbf{i}, 1, 0, 0, 1, 0)^T$ and $\varphi(v_3) = \varphi(v_2)^*$, and

$$v_2 = \begin{pmatrix} \mathbf{i} \\ 1 - \mathbf{j} \\ 0 \end{pmatrix}.$$

By Lemma 6.5 and $\phi(A)\varphi(u) = \varphi(v_2) + \varphi(u)\lambda$, we can get $\varphi(u) = (1 - \mathbf{i}, -1 - 2\mathbf{i}, -\mathbf{i}, 0, 0, -\mathbf{i})^T$ and

$$u = \begin{pmatrix} 1 - \mathbf{i} \\ -1 - 2\mathbf{i} \\ -\mathbf{i} - \mathbf{k} \end{pmatrix}.$$

For any solutions $\exp\{At\}\eta$ of Eq.(6.18), substituting v_1, v_2, u into Eq.(6.9), let $\eta = v_1 r_1 + v_2 r_2 + u r_3$. We can get

$$\exp\{At\}\eta = v_1 e^{it} r_1 + v_2 e^t r_2 + (u + v_2 t) e^t r_3$$

namely

$$\exp\{At\}\eta = \begin{pmatrix} -\mathbf{i} - \mathbf{j} & \mathbf{i} & 1 - \mathbf{i} + it \\ 0 & 1 - \mathbf{j} & 1 - 2\mathbf{i} - (\mathbf{i} + \mathbf{k})t \\ 0 & 0 & -\mathbf{i} - \mathbf{k} \end{pmatrix} \begin{pmatrix} e^{it} r_1 \\ e^t r_2 \\ e^t r_3 \end{pmatrix}.$$

Let $\eta = (1, 0, 0)^T, (0, 1, 0)^T, (0, 1, 0)^T$ in turn, we can obtain three linear independent solutions, which compose the fundamental matrix $\exp\{At\}$, namely

$$\exp\{At\} = \begin{pmatrix} \alpha \frac{\mathbf{i} + \mathbf{j}}{2} & \alpha \frac{1 - \mathbf{i} + \mathbf{j} + \mathbf{k}}{4} + \frac{\mathbf{i} + \mathbf{k}}{2} e^t & \alpha \frac{2 + \mathbf{i} - \mathbf{j}}{2} + \left(\frac{\mathbf{i} + \mathbf{j} - \mathbf{k} - t - \mathbf{j}t}{2} \right) e^t \\ 0 & e^t & \mathbf{k} t e^t \\ 0 & 0 & e^t \end{pmatrix}.$$

where $\alpha = -(\mathbf{i} + \mathbf{j})e^{it}$.

7 Conclusion

In this paper, we presented an algorithm to evaluate the fundamental matrix by employing the eigenvalue and eigenvectors. We gave a method to construct the fundamental matrix when the linear system has multiple eigenvalues. In particular, if the number of independent eigenvectors might be smaller than the dimensionality of the system. That is, the numbers of the eigenvectors is not enough to construct a fundamental matrix. We therefore have to discover how to find the “missing solutions”. The main purpose is to answer this question.

8 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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